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Output Tracking with Prescribed Transient Behaviour for Linear Systems with Input Hysteresis

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Abstract. Tracking of a reference signal (assumed bounded with essentially bounded derivative) is considered for single-input, single-output, linear, minimum-phase systems of relative degree one, with a hysteresis nonlinearity in the input channel. The first control objective is tracking, by the output y, with prescribed accuracy: given $\lambda > 0$ (arbitrarily small), determine a feedback strategy which ensures that, for every reference signal r, the tracking error $e = y - r$ is ultimately bounded by λ (that is, $|e(t)| < \lambda$ for all t sufficiently large). The second objective is guaranteed output transient performance: the evolution of the tracking error e should be contained in a prescribed performance funnel $\mathcal{F}_\beta$ (determined by a function β). Under mild assumptions on the hysteresis operator, both objectives are achieved by a non-adaptive memoryless feedback of the form $u(t) = \nu(k(t))e(t)$, where ν is any continuous function with properties $\liminf_{\kappa\to\infty} \nu(\kappa) = -\infty$ and $\limsup_{\kappa\to\infty} \nu(\kappa) = +\infty$, and k is generated via a feedback function of the tracking error and the funnel parameter β.

1 Introduction

1.1 Linear systems with input hysteresis

Let $\mathcal{L}$ denote the class of finite-dimensional, real, single-input, single-output, linear, minimum-phase systems (A, b, c) of relative degree one. The relative degree one property corresponds to the condition $cb \neq 0$ and the minimum-phase property is characterized by

$$\det \begin{bmatrix} sI - A & b \\ c & 0 \end{bmatrix} \neq 0 \quad \text{for all } s \in \mathbb{C}_+ := \{s \in \mathbb{C} \mid \operatorname{Re}(s) \geq 0\}. \tag{1}$$

Specifically,

$$\mathcal{L} = \{(A,b,c) |\ A \in \mathbb{R}^{n\times n},\ b, c^T \in \mathbb{R}^n,\ n \in \mathbb{N},\ cb \neq 0,\ (1) \text{ holds}\}.$$

Before introducing the class of hysteresis nonlinearities, we assemble some notation and terminology. Let $\mathbb{R}_+ := [0,\infty)$, $I \subset \mathbb{R}_+$ (an interval) and $\mathcal{D} \subset \mathbb{R}^q$. We denote the space of continuous functions $I \to \mathcal{D}$ by $C(I,\mathcal{D})$: if $\mathcal{D} = \mathbb{R}$, then we simply write $C(I)$. Let $\alpha \geq 0$. For $w \in C([0,\alpha])$ and $\gamma, \delta > 0$, we define

$$\mathcal{C}(w;\delta,\gamma) := \left\{ v \in C([0,\alpha+\gamma]) \big|\ v|_{[0,\alpha]} = w,\ \|v(t) - w(t)\|_{C([0,\alpha+\gamma])} \leq \delta \right\}.$$

A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a *time transformation* if it is continuous, non-decreasing and surjective. We denote the set of all time transformations by $\mathcal{T}$. An operator $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is *rate independent* if

$$(\Phi(u \circ f))(t) = (\Phi u)(f(t)) \quad \forall\, u \in C(\mathbb{R}_+), \quad \forall\, f \in \mathcal{T}, \quad \forall\, t \in \mathbb{R}_+.$$

We say that $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is a *hysteresis operator* if Φ is causal and rate independent. We consider hysteresis operators that satisfy the following assumptions:

(A1) for all $\alpha \geq 0$ and all $w \in C([0,\alpha])$, there exist constants $\delta, \gamma, \theta > 0$ such that

$$\|\Phi(v_1) - \Phi(v_2)\|_{C([0,\alpha+\gamma])} \leq \theta\, \|v_1 - v_2\|_{C([0,\alpha+\gamma])} \quad \forall\, v_1, v_2 \in \mathcal{C}(w;\delta,\gamma);$$

(A2) for all $\alpha > 0$ and all $u \in C([0,\alpha))$, there exists $\gamma > 0$ such that

$$\sup_{t\in[0,\tau]} |(\Phi u)(t)| \leq \gamma \left(1 + \sup_{t\in[0,\tau]} |u(t)|\right) \quad \forall\, \tau \in [0,\alpha);$$

(A3) there exist δ, $\Delta > 0$ such that, for all $u \in C(\mathbb{R}_+)$,

$$|u(t)| > \Delta \quad \Longrightarrow \quad \delta u^2(t) \leq u(t)(\Phi u)(t);$$

and denote this set of hysteresis operators by

$$\mathcal{H} := \{\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+) |\ \Phi \text{ causal, rate independent, (A1), (A2) \& (A3) hold}\}.$$

The technical assumptions (A1) and (A2) underpin the existence result in Theorem 2.1 below. Assumption (A3) is a weak sector-bounded condition. Many of the hysteretic effects encountered in practice are of class $\mathcal{H}$. One such example backlash hysteresis.

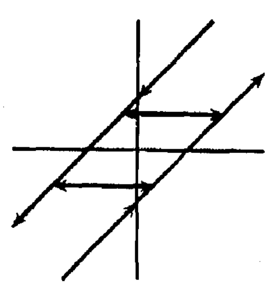

Fig.1. Backlash hysteresis

More complex hysteretic elements, such as Prandtl and Preisach operators (see, e.g., [1, 4]) are also encompassed within our framework.

The main concern of the paper is a study of series connections of a hysteresis operator $\Phi \in \mathcal{H}$ and a linear system $(A, b, c) \in \mathcal{L}$, that is, systems of the form

$$\dot{x}(t) = Ax(t) + b(\Phi u)(t), \quad x(0) = x^0 \in \mathbb{R}^n, \quad y(t) = cx(t) \in \mathbb{R}. \tag{2}$$

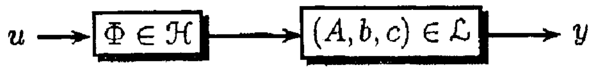

Fig. 2.

1.2 Control objectives and the performance funnel

The first control objective is approximate tracking, by the output y, of reference signals r of class $\mathcal{R} := W^{1,\infty}(\mathbb{R}_+)$, i.e. the space of locally absolutely continuous bounded functions $\mathbb{R}_+ \to \mathbb{R}$ with bounded derivative. In particular, for arbitrary $\lambda > 0$, we seek an output feedback strategy which ensures that, for every $r \in \mathcal{R}$, the closed-loop system has bounded solution and the tracking error $e(t) = y(t) - r(t)$ is ultimately bounded by λ (that is, $|e(t)| < \lambda$ for all t sufficiently large). The second control objective is prescribed transient behaviour of the tracking error signal. We capture both objectives in the concept of a performance funnel, introduced in [3],

$$\mathcal{F}_\beta := \left\{ (t, e) \in \mathbb{R}_+ \times \mathbb{R}^m \middle| \ \beta(t)\,|e| < 1 \right\}$$

associated with a function $\beta : \mathbb{R}_+ \to \mathbb{R}$ (the reciprocal of which determines the funnel boundary) belonging to

$$\mathcal{B}_\lambda := \left\{ \beta \in W^{1,\infty}(\mathbb{R}_+) \middle| \ \beta(0) = 0,\ \beta(s) > 0 \,\forall\, s > 0,\ \liminf_{s \to \infty} \beta(s) \geq 1/\lambda \right\}.$$

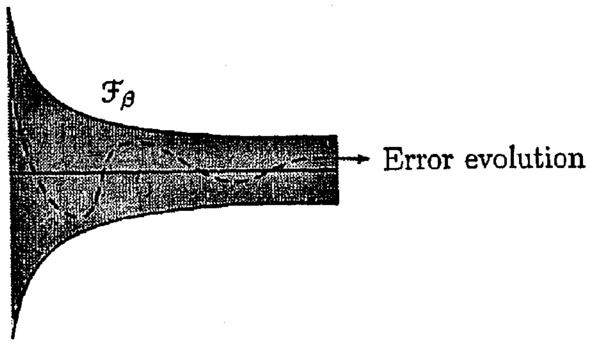

Fig. 3. Prescribed performance funnel $\mathcal{F}_\beta$.

The aim is an output feedback strategy ensuring that, for every reference signal $r \in \mathcal{R}$, the tracking error $e = y - r$ evolves within the funnel $\mathcal{F}_\beta$ and all signals are bounded. For every $\lambda > 0$ and $\beta \in \mathcal{B}_\lambda$, evolution within the funnel ensures that the first control objective is achieved; if β is chosen as the function $t \mapsto \min\{t/T, 1\}/\lambda$, then evolution within the funnel ensures that the prescribed tracking accuracy $\lambda > 0$ is achieved within the prescribed time $T > 0$.

1.3 Output feedback

Let $\nu\colon \mathbb{R} \to \mathbb{R}$ be any continuously differentiable function with the properties

$$\limsup_{k\to\infty} \nu(k) = +\infty \quad \text{and} \quad \liminf_{k\to\infty} \nu(k) = -\infty. \tag{3}$$

A simple example of a function satisfying (3) is $\nu : k \mapsto k \cos k$. For arbitrary $r \in \mathcal{R}$, $\lambda > 0$ and $\beta \in \mathcal{B}_\lambda$, consider the control strategy

$$u(t) = \nu(k(t))\,[y(t) - r(t)], \qquad k(t) = \frac{1}{1 - \big(\beta(t)\,[y(t) - r(t)]\big)^2}. \tag{4}$$

Control of systems with input hysteresis has also been considered in [7]: therein, two-sided sector bounds, stronger than (A3), are imposed on Φ; the admissible class of functions ν is more restrictive than that of the present paper; furthermore, the gain k is determined adaptively, viz., with $\lambda > 0$,

$$\dot{k}(t) \;=\; \max\{|y(t) - r(t)| - \lambda,\ 0\}\ |y(t) - r(t)|.$$

This gain adaptation (and variations thereof), which generates a monotone gain function, is typical in the area of high-gain adaptive control initiated by [6], [2], and [5]. By contrast, the approach of the present paper is non-adaptive: the gain k is not monotone and the controller is memoryless.

The main contribution of the paper is to show that the feedback (4) applied to (2) achieves the control objectives. Noting the potential singularity in (4), care must be exercised in interpreting the closed-loop system. This we do in the next section.

2 The closed-loop system

Let $r \in \mathcal{R}$, $\lambda > 0$, $\beta \in \mathcal{B}_\lambda$ and let $\nu\colon \mathbb{R} \to \mathbb{R}$ be continuously differentiable with properties (3). The conjunction of the system (2) and control (4) yields the closed-loop initial-value problem

$$\left.\begin{aligned} &\dot{x}(t) = Ax(t) + b(\Phi(u))(t), \quad x(0) = x^0 \in \mathbb{R}^n \\ &u(t) = \nu(k(t))\big(cx(t) - r(t)\big) \\ &k(t) = 1/\big(1 - [\beta(t)(cx(t) - r(t))]^2\big). \end{aligned}\right\} \tag{5}$$

By a *solution* of (5), we mean a continuously differentiable function $x : [0, \tau) \to \mathbb{R}^n$, with $x(0) = x^0$, satisfying the differential equation in (5) and such that $\beta(t)|cx(t) - r(t)| < 1$

for all $t \in [0, \tau)$. A solution is *maximal* if it has no proper right extension that is also a solution. In developing an existence (and uniqueness) theory for this highly nonlinear system, we will have occasion to give meaning to $\Phi(u)$, where $u \in C([0, \tau))$ with τ possibly finite. To this end, for each $\alpha \in (0, \tau)$, define the function $u_\alpha \in C(\mathbb{R}_+)$ by

$$u_\alpha(t) := \begin{cases} u(t), & t \in [0, \alpha] \\ u(\alpha), & t > \alpha . \end{cases}$$

Then we interpret $\Phi(u)$ as the unique function $f \in C([0, \tau))$ with the property $\Phi(v_\alpha)|_{[0,\alpha]} = f|_{[0,\alpha]}$ for all $\alpha \in (0, \tau)$.

Writing

$$\mathcal{D} := \{(x, w) \in \mathbb{R}^n \times \mathbb{R} | \ \beta(|w|)|cx - r(|w|)| < 1\}, \tag{6}$$

and introducing $\varphi : C(\mathbb{R}_+, \mathcal{D}) \to C(\mathbb{R}_+, \mathbb{R})$ given by

$$(\varphi(p, q))(t) := \nu\big((1 - [\beta(|q(t)|)(cp(t) - r(|q(t)|)]^2)^{-1}\big) \left(cp(t) - r(|q(t)|)\right),$$

the closed-loop system (5) can be re-interpreted as the following initial-value problem for an autonomous functional differential equation

$$\left.\begin{aligned} &\dot{x}(t) = Ax(t) + b(\Phi(\varphi(x, w)))(t), \\ &\dot{w}(t) = 1, \\ &(x(0), w(0)) = (x^0, 0) \in \mathcal{D} . \end{aligned}\right\} \tag{7}$$

Clearly, x is a solution (respectively, maximal solution) of (5) if, and only if, $t \mapsto (x(t), t)$ is a solution (respectively, maximal solution) of (7).

Invoking properties (A1) and (A2) of $\Phi \in \mathcal{H}$, and observing that φ is locally Lipschitz, the arguments leading to [4, Theorem 2.2] are readily modified to yield the following existence and uniqueness result.

Theorem 2.1 *For each x^0, the initial-value problem* (7) *has unique maximal solution $(x, w) : [0, \tau) \to \mathcal{D}$. Moreover, if the closure of the orbit $(x, w)([0, \tau))$ is a compact subset of $\mathcal{D}$, then $\tau = \infty$.* •

We are now in a position to prove the main result of the paper.

Theorem 2.2 *Let $(A, b, c) \in \mathcal{L}$ and let $\Phi \in \mathcal{H}$ be such that (A3) holds. Let $r \in \mathcal{R}$, $\lambda > 0$ and $\beta \in \mathcal{B}_\lambda$. For each $x^0 \in \mathbb{R}^n$, the closed-loop initial-value problem* (5) *has unique solution $x : \mathbb{R}_+ \to \mathbb{R}^n$. Moreover,*

(i) *there exists $\varepsilon \in (0, 1)$ such that, for all $t \in \mathbb{R}_+$, $\beta(t)\,|cx(t) - r(t)| \le 1 - \varepsilon$;*

(ii) *the continuous functions $u, \Phi(u) : \mathbb{R}_+ \to \mathbb{R}$ and $k : [1, \infty) \to \mathbb{R}_+$ are bounded.*

Sketch proof.

STEP 1: An application of Theorem 2.1 establishes the existence of a unique maximal solution $x: [0,\tau) \to \mathbb{R}$ of (5), with $0 < \tau \le \infty$. Observe that, by the properties of $\mathcal{B}_\lambda$ and since the solution evolves in the funnel $\mathcal{F}_\beta$, the function $cx - r$ is bounded and so, by boundedness of r, $y = cx$ is also bounded.

STEP 2: We will establish the existence of $c_1 > 0$ such that

$$\frac{d}{dt}\big(\beta(t)e(t)\big)^2 \le c_1 + 2cb\,\beta^2(t)e(t)\big(\Phi(\nu(k)e)\big)(t) \quad \text{for a.a. } t \in [0,\tau). \tag{8}$$

By the minimum-phase and relative-degree-one properties of $(A,b,c) \in \mathcal{L}$, there exists a linear coordinate transformation $L : x \mapsto (cx, z)$ which, on writing $(y(t), z(t)) = Lx(t)$, takes (2) into the form

$$\begin{aligned}
\dot y(t) &= A_1 y(t) + A_2 z(t) + cb(\Phi(u))(t),\\
\dot z(t) &= A_3 y(t) + A_4 z(t),\\
(y(0), z(0)) &= (y^0, z^0) = Lx^0.
\end{aligned}$$

By the minimum-phase property, $\sigma(A_4) \subset \mathbb{C}_+$ and so, by boundedness of y, we may immediately infer boundedness of z, whence boundedness of x. Writing

$$e(t) = y(t) - r(t),$$

we have

$$\begin{aligned}
&\dot e(t) = A_1 e(t) + A_2 z(t) + f_1(t) + cb(\Phi(u))(t) \qquad \text{for a.a. } t \in [0,\tau),\\
&\dot z(t) = A_3 e(t) + A_4 z(t) + f_2(t), \qquad \text{for all } t \in [0,\tau)\\
&u(t) = \nu(k(t))e(t), \qquad k(t) = 1/(1 - (\beta(t)e(t))^2)\\
&(y(0), z(0)) = (y^0, z^0) = Lx^0,
\end{aligned}$$

where the essentially bounded function f_1 and the bounded function f_2 are given by

$$f_1(t) := A_1 r(t) - \dot r(t), \quad f_2(t) := A_3 r(t).$$

We may now conclude the existence of $c_0 > 0$ such that

$$\dot e(t) = c_0 + cb\big(\Phi(\nu(k)e)\big)(t) \text{ for a.a. } t \in [0,\tau). \tag{9}$$

Now,

$$\frac{d}{dt}\big(\beta(t)e(t)\big)^2 = 2\beta(t)\dot\beta(t)e^2(t) + 2\beta^2(t)e(t)\dot e(t)$$

and so, by boundedness of e, β and essential boundedness of its derivative $\dot\beta$, together with (9), there exists $c_1 > 0$ such that (8) holds.

STEP 3: Next, we show that k is bounded. Seeking a contradiction, suppose that k is unbounded. By properties (3) of ν, there exists a strictly increasing unbounded sequence

(k_j) in $(1,\infty)$ such that $cb\,\nu(k_j)<0$ for all $j\in\mathbb{N}$ and $\lim_{j\to\infty} cb\,\nu(k_j)=-\infty$. For each $j\in\mathbb{N}$, define

$$\begin{aligned}
\tau_j &:= \inf\{t\in[0,\tau)\mid k(t)=k_{j+1}\}\\
\sigma_j &:= \sup\{t\in[0,\tau_j]\mid cb\,\nu(k(t))=cb\,\nu(k_j)\}\\
\tilde{\sigma}_j &:= \sup\{t\in[0,\tau_j]\mid k(t)=k_j\}\le\sigma_j\,.
\end{aligned}$$

Then,

$$\left.\begin{array}{l}
k_j\le k(t) \quad\text{and}\quad |\nu(k_j)|\le|\nu(k(t))|\\
(\beta(t)e(t))^2=1-(1/k(t))\ge 1-(1/k_j)\ge 1-(1/k_1)=:c_2\\
|e(t)|\ge c_3:=\sqrt{c_2}/\big(\sup_{s\ge0}\beta(s)\big)
\end{array}\right\}\quad \forall\, t\in[\sigma_j,\tau_j]\quad\forall\, j\in\mathbb{N}. \tag{10}$$

In view of (8) and recalling that $cb\,\nu(k_j)<0$ for all j, we have

$$\frac{d}{dt}\big(\beta(t)e(t)\big)^2\le c_1-\frac{2|cb|\beta^2(t)}{|\nu(k(t))|}\,\nu(k(t))e(t)\big(\Phi(\nu(k)e)\big)(t)\quad\text{a.a. } t\in[\sigma_j,\tau_j]\ \ \forall\, j\in\mathbb{N}. \tag{11}$$

By assumption (A3) on Φ, there exist $\delta,\Delta>0$ such that

$$t\in[0,\tau),\ |u(t)|\ge\Delta \quad\Longrightarrow\quad \delta u^2(t)\le u(t)(\Phi(u))(t).$$

Choose $j^*\in\mathbb{N}$ sufficiently large so that

$$c_3|\nu(k_{j^*})|\ge\Delta \quad\text{and}\quad c_1-2c_2\delta|cb||\nu(k_{j^*})|<0.$$

By (10), (11) and invoking (A3), we have

$$\begin{aligned}
\frac{d}{dt}\big(\beta(t)e(t)\big)^2 &\le c_1-\frac{2|cb|\beta^2(t)}{|\nu(k(t))|}\,\delta\big(\nu(k(t)e(t)\big)^2\\
&= c_1-2\delta|cb||\nu(k_{j^*})|\big(\beta(t)e(t)\big)^2\\
&\le c_1-2c_2\delta|cb||\nu(k_{j^*})|<0\quad\forall\, t\in[\sigma_{j^*},\tau_{j^*}].
\end{aligned} \tag{12}$$

Therefore

$$\big(\beta(\tau_{j^*})e(\tau_{j^*})\big)^2-\big(\beta(\sigma_{j^*})e(\sigma_{j^*})\big)^2<0,$$

whence the contradiction

$$0>\frac{1}{1-\big(\beta(\tau_{j^*})e(\tau_{j^*})\big)^2}-\frac{1}{1-\big(\beta(\sigma_{j^*})e(\sigma_{j^*})\big)^2}=k(\tau_{j^*})-k(\sigma_{j^*})\ge0.$$

This proves boundedness of k.

STEP 4: Boundedness of e and k implies boundedness of $u=\nu(k)e$. Furthermore, since k is bounded, there exists $\varepsilon>0$ such that $\beta(t)|e(t)|\le1-\varepsilon$ for all $t\in[0,\tau)$.

STEP 5: Next, we prove that $\tau = \infty$. Suppose that τ is finite. Let $c_4 > 0$ be such that $|cx(t)| \le c_4$ for all $t \in [0, \tau)$, and define

$$\mathcal{C} := \{(x, w) \in \mathcal{D} | \ \beta(|w|)\,|cx| \le 1 - \varepsilon, \ \ |cx| \le c_4, \ \ w \in [0, \tau]\}$$

where $\mathcal{D}$ is given by (6). Then $\mathcal{C}$ is a compact subset of $\mathcal{D}$ and contains the maximal solution $t \mapsto (x(t), t)$ of (7) and so, by Theorem 2.1, τ cannot be finite - a contradiction. Therefore, the supposition that τ is finite is false. It remains only to show that $\Phi(u)$ is bounded: this is an immediate consequence of boundedness of u and (A3). This completes the proof. □

Remark. If $(A, b, c) \in \mathcal{L}$ is such that the sign of the high-frequency gain $cb \neq 0$ is known, then the need for a function ν with properties (3) is obviated. In particular, if $cb > 0$, then the first of equations (4) can be replaced by $u(t) = -k(t)[y(t) - r(t)]$.

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